Problem with solution proposed by Arkady Alt , San Jose , California, USA .

Prove that for any real $p > 1$ and $x > 1$ holds inequality $\dfrac{\ln x}{\ln(x+p)} \leq \left(\dfrac{\ln(x+p-1)}{\ln(x+p)}\right)^p$.

Solution.

Note that $\dfrac{\ln x}{\ln(x+p)} \leq \left(\dfrac{\ln(x+p-1)}{\ln(x+p)}\right)^p \Leftrightarrow \ln x \cdot \ln^{p-1}(x+p) \leq \ln^p(x+p-1)$.

Since $\ln\ln x$ is concave down on (e^{-1},∞) then for any $x_1,x_2,\ldots,x_n \in (1,\infty)$ and positive weights $p_1,p_2,\ldots,p_n$ such that $p_1+p_2+\ldots+p_n = 1$ holds wighted Jensen's nequality

$p_1\ln\ln x_1 + p_2\ln\ln x_2 + \ldots + p_n\ln\ln x_n \leq \ln\ln(p_1x_1+p_2x_2+\ldots+p_nx_n) \Leftrightarrow$

(1) $\ln^{p_1}x_1 \cdot \ln^{p_2}x_2 \cdot \ln^{p_n}x_n \leq \ln(p_1x_1+p_2x_2+\ldots+p_nx_n)$.

In particular for $x_1 = x, x_2 = x+p, p_1 = \dfrac{1}{p}, p_2 = \dfrac{p-1}{p}$ we have

$\ln^{\frac{1}{p}}x \cdot \ln^{\frac{p-1}{p}}(x+p) \leq \ln\left(\dfrac{x}{p} + \dfrac{(p-1)(x+p)}{p}\right) = \ln\left(\dfrac{x+p^2-p-x+px}{p}\right) = \ln(x+p-1) \Leftrightarrow$

(2) $\ln x \cdot \ln^{p-1}(x+p) \leq \ln^p(x+p-1)$.

Using inequality (2) we obtain

$\prod_{k=2}^{n+1}\dfrac{\ln k}{\ln(k+p)} \leq \prod_{k=2}^{n+1}\left(\dfrac{\ln(k+p-1)}{\ln(k+p)}\right)^p = \left(\dfrac{\ln(2+p-1)}{\ln(n+1+p)}\right)^p = \dfrac{\ln^p(1+p)}{\ln^p(n+1+p)}$.

Hence, $\sum_{n=1}^{\infty}\dfrac{1}{n}\cdot\prod_{k=2}^{n+1}\dfrac{\ln k}{\ln(k+p)} \leq \ln^p(1+p)\sum_{n=1}^{\infty}\dfrac{1}{n\ln^p(n+1+p)} < \ln^p(1+p)\sum_{n=1}^{\infty}\dfrac{1}{n\ln^p n}$,

where latter series is convergent.

Remark.

Inequality (1) and (2) can be proved by elementary way for any rational weights.

First we will prove one auxiliary inequality

(3) $\ln(x-h)\ln(x+h) \leq \ln^2 x, x > 0$ and $0 \leq h < x$.

Proof 1.

$$\ln(x-h)\ln(x+h) \leq \left(\frac{\ln(x-h)+\ln(x+h)}{2}\right)^2 = \left(\frac{\ln(x^2-h^2)}{2}\right)^2 \leq \left(\frac{\ln(x^2)}{2}\right)^2 = \ln^2 x.$$

Proof 2.

$\ln(x-h)\ln(x+h) = \left(\ln x + \ln\left(1-\dfrac{h}{x}\right)\right)\left(\ln x + \ln\left(1+\dfrac{h}{x}\right)\right) = \ln^2 x + \ln x \cdot \ln\left(1-\dfrac{h^2}{x^2}\right) +$
$\ln\left(1-\dfrac{h}{x}\right)\cdot\ln\left(1+\dfrac{h}{x}\right) \leq \ln^2 x$ since $\ln\left(1-\dfrac{h^2}{x^2}\right) \leq 0$ and $\ln\left(1-\dfrac{h}{x}\right) \leq 0$.

From inequality (3) follows that for any $x_1,x_2 > 1$ holds inequality

(4) $\ln x_1 \ln x_2 \leq \ln^2\dfrac{x_1+x_2}{2}$.

Indeed, by setting $x := \dfrac{x_1+x_2}{2}$ and $h := \dfrac{|x_2-x_1|}{2}$ inequality (4) becomes (3).

Suppose that inequality $\ln x_1\ldots\ln x_k \leq \ln^k\dfrac{x_1+\ldots+x_k}{k}$ holds for any $2 \leq k < n$.

For $n = 2m$ inequality $\ln x_1\ldots\ln x_n \leq \ln^n\dfrac{x_1+\ldots+x_n}{n}$ can be obtained by doubling procedure.

If $n = 2m-1$ and $x_1,\ldots,x_{2m-1} > 1$ then for $n+1 = 2m$ numbers $x_1,\ldots,x_n,x_{n+1}$ where $x_{n+1} := \dfrac{x_1+\ldots+x_n}{n}$ we have $\ln x_1\ldots\ln x_n\ln x_{n+1} \leq \ln^{n+1}\dfrac{x_1+\ldots+x_n+x_{n+1}}{n+1} =$

$\ln^{n+1}\frac{nx_{n+1}+x_{n+1}}{n+1} = \ln^{n+1}x_{n+1} \Leftrightarrow \ln x_1\ldots\ln x_n \le \ln^n x_{n+1}$.

Let $x_1 =..= x_m = a, x_{m+1} =\ldots= x_n = b$ then we obtain inequality

$\ln^m a \bullet \ln^{n-m} b \le \ln^n \frac{ma+(n-m)b}{n} \Leftrightarrow \ln^p a \bullet \ln^q b \le \ln^n(pa+qb),$

where $p := \frac{m}{n}, q := \frac{n-m}{n}$.

Generalization.

Let f be logarithmicly concave down on interval I function.Then for any $x \in I$ and any positive

$p > 1$, such that $x+p \in I$, holds inequality

(5) $$\frac{f(x)}{f(x+p)} \le \left(\frac{f(x+p-1)}{f(x+p)}\right)^p.$$

First recall definition of logarithmicly concave function.

Definition.

Function f is logarithmicly concave down on interval I if f is positive on I and $\ln f$ is concave down on I.

Thus, for any $x_1, x_2 \in I$ we have inequality $\frac{\ln f(x_1)+\ln f(x_2)}{2} \ge \ln f\left(\frac{x_1+x_2}{2}\right) \Leftrightarrow$.

$f(x_1)\bullet f(x_2) \ge f^2\left(\frac{x_1+x_2}{2}\right)$.

Further by the same procedure we obtain regular Jensen Inequality

$f(x_1)\bullet f(x_2)\bullet\ldots\bullet f(x_n) \ge f^n\left(\frac{x_1+x_2+\ldots+x_n}{n}\right), x_1, x_2, \ldots, x_2 \in I$ and, further, weighted Jensen's Inequality with positive rational weights $p_1, p_2, \ldots, p_n$:

$f^{p_1}(x_1)\bullet f^{p_2}(x_2)\bullet\ldots\bullet f^{p_n}(x_n) \ge f^{p_1+p_2+\ldots+p_n}\left(\frac{p_1x_1+p_2x_2+\ldots+p_nx_n}{p_1+p_2+\ldots+p_n}\right)$.

Using limit representation of real numbers we extend this inequality on real weights.

In particular for $x_1 = x, x_2 = x+p, x \in I, p > 1,\ x+p \in I$ and weights $\frac{1}{p}, \frac{p-1}{p}$ we obtain inequality (5).